Inverse scattering problems for nonlinear equations on Lorentzian manifolds

Teemu Tyni

AIP 2023
Göttingen

Joint work with
Spyros Alexakis, Matti Lassas, and Hiroshi Isozaki

University of Oulu

Brief history of nonlinear hyperbolic problems

- Kurylev, Lassas, Uhlmann (2014-2018): Using nonlinearity and higher order linearization to solve inverse problems
- Since then, techniques using nonlinearity as a tool have been extremely popular: T Balehowsky, C Cârstea, X Chen, M de Hoop, A Feizmohammadi, C Guillarmou, P Hintz, Y Kian, H Koch, K Krupchyk, M Lassas, T Liimatainen, Y-H Lin, G Nakamura, L Oksanen, G Paternain, A Rüland, M Salo, P Stefanov, G Uhlmann, Y Wang, J Zhai, and many more (apologies to any who I missed!)

Motivation for the scattering problem

Can one recover the spacetime structure from scattering data?

A couple of examples

- Sá Barreto (Duke 2005): if space part of spacetime is asymptotically hyperbolic and setting is time-independent (static), then Carleman estimates and boundary control yield unique determination of metric: $\Phi^* g_1 = g_2$
- Sá Barreto, Wang, Uhlmann (arxiv 2021): nonlinear scattering. Potential recovery for $(\partial_t^2 - \Delta)u + f(u)$, $f(u) \sim u^5$, via Melrose-type compactification (stereographic projection to compactify space)

Penrose conformal compactification

Let $(\mathbb{R}^{1+3}, \eta)$ be the Minkowski space with its standard metric η in polar coordinates (t, r, θ, ϕ):

$$\eta = -dt^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2\right)$$

We make a conformal change to $\tilde{\eta} := \Omega^2\eta$ with

$$\Omega = 4\frac{1}{1+(t+r)^2}\frac{1}{1+(t-r)^2}$$

Penrose conformal compactification

- Let $\tilde{\eta} = \Omega^2 \eta$, where

$$\Omega = 4 \frac{1}{1+(t+r)^2} \frac{1}{1+(t-r)^2}$$

- $\Phi : \mathbb{R}^{1+3} \to \mathbb{R} \times \mathbb{S}^3$, defined by

$$\Phi(t, r, \theta, \varphi) = (T, R, \theta, \varphi),$$

where

$$\begin{aligned} T &= \arctan(t+r) + \arctan(t-r), \\ R &= \arctan(t+r) - \arctan(t-r), \\ -\pi < T + R < \pi, &\quad -\pi < T - R < \pi, \quad R \geq 0. \end{aligned}$$

On $\mathbb{S}^3$ we have the standard spherical coordinates (R, θ, φ), and the metric on the cylinder $\mathbb{R} \times \mathbb{S}^3$ is of the form

$$g_{\mathbb{R}\times\mathbb{S}^3} = -dT^2 + dR^2 + \sin^2(R)\left(d\theta^2 + \sin^2(\theta) d\varphi^2\right).$$

Penrose conformal compactification

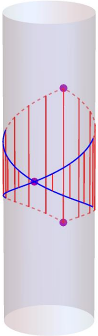
Thus

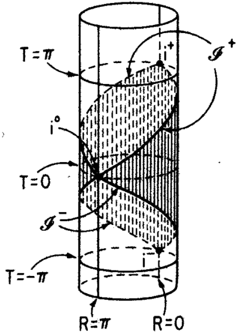
$$\Phi : \mathbb{R}^{1+3} \to \mathbb{R} \times \mathbb{S}^3$$

is conformal diffeomorphism. We call

$$\widehat{N} = \Phi(\mathbb{R}^{1+3}) \subset \mathbb{R} \times \mathbb{S}^3$$

the **Penrose diagram** of $\mathbb{R}^{1+3}$ and Φ the Penrose map.





Right picture: R. Wald *General relativity*, 1984

Penrose conformal compactification

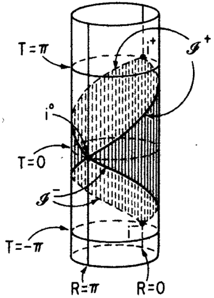
Thus

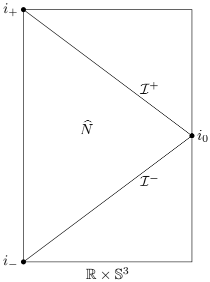
$$\Phi : \mathbb{R}^{1+3} \to \mathbb{R} \times \mathbb{S}^3$$

is conformal (isometric) diffeomorphism. We call

$$\widehat{N} = \Phi(\mathbb{R}^{1+3}) \subset \mathbb{R}\times\mathbb{S}^3$$

the Penrose diagram of $\mathbb{R}^{1+3}$.





Notation for the wave equation

Let $(\mathbb{R}^{1+3}, g)$ be a globally hyperbolic Lorentzian manifold and consider the nonlinear wave equation

$$\Box_g u(t,y) + a(t,y)u(t,y)^\kappa = 0, \quad (t,y) \in \mathbb{R}^{1+3}.$$

where $\kappa \geq 4$ is an integer and $a(t,y)$ a smooth rapidly decaying function.
Here $\Box_g$ is the D'Alembertian wave operator

$$\Box_g u = -\sum_{a,b=0}^{n} \frac{1}{\sqrt{|\det(g)|}} \frac{\partial}{\partial x^a} \left(\sqrt{|\det(g)|} g^{ab} \frac{\partial u}{\partial x^b} \right)$$

Towards a scattering problem

- Let η be the standard Minkowski metric on $\mathbb{R}^{n+1}$.
- Let g be a globally hyperbolic Lorentzian metric on $\mathbb{R}^{n+1}$, such that $g_{ij}(x) - \eta_{ij}$ is a Schwartz rapidly decaying function and
- Let $\tilde{g} = \Omega^2 g$ be a conformal metric to g and let $\widehat{g} = \Phi_* \tilde{g}$ be the pushforward metric on the Penrose diagram.

Then u satisfies the nonlinear wave equation

$$\Box_g u + a u^\kappa = 0$$

iff $\tilde{u} = (\Omega^{-1} u) \circ \Phi^{-1}$ satisfies

$$(\Box_{\widehat{g}} + B)\tilde{u} + A\tilde{u}^\kappa = 0$$

in $\widehat{N}$, where

$$A := (\Phi^{-1})^*(a\Omega), \quad B := \frac{1}{6}(\Phi^{-1})^*(R_{\Omega^2 g} - \Omega^{-2} R_g).$$

A geometric scattering problem

We say that a function $u \in H^m_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^3)$ is a solution of the scattering problem on $(\mathbb{R} \times \mathbb{R}^3, g)$, with the past radiation field h_-,

$$\begin{cases} \Box_g u(x) + a(x) \cdot u(x)^k = 0, & \text{in } \mathbb{R} \times \mathbb{R}^3, \\ u(x) \sim h_-(x) & \text{as } x \text{ goes to } \mathcal{I}^- \end{cases}$$

if the function $\tilde{u} = (\Omega^{-1} u) \circ \Phi^{-1}$ satisfies $\tilde{u} \in H^m(\widehat{N})$ and it is a solution of the Goursat-Cauchy boundary value problem

$$\begin{cases} \Box_{\widehat{g}} \tilde{u}(x) + B(x)\tilde{u}(x) + A(x) \cdot \tilde{u}(x)^k = 0, & \text{in } \widehat{N}, \\ \tilde{u}|_{\mathcal{I}^-} = \tilde{h}_-, \end{cases}$$

Scattering problem has a solution

Lemma

Let g be a globally hyperbolic Lorentzian metric on $\mathbb{R}^{3+1}$, where $g-\eta$ is in the Schwartz class. Let $(\widehat{N},\Omega^2 g)$ be the Penrose diagram. Let $-\pi < T_- < 0 < T_+ < \pi$. There is $\varepsilon > 0$ such that the following holds: Let $h \in H^m(\mathcal{I}^-)$ be such that $\operatorname{supp}(h) \subset \{x \in \mathcal{I}^- \mid \mathrm{t}(x) > T_-\}$ and $\|h\|_{H^m(\mathcal{I}^-)} < \varepsilon$ (m large enough). Then the non-linear scattering problem

$$\begin{cases} (\Box_g + B)u + Au^k = 0, \quad \text{in } \{x \in \widehat{N} : \ \mathrm{t}(x) < T_+\}, \\ u\big|_{\mathcal{I}^-} = h, \\ u = 0 \text{ in } \{x \in \widehat{N} : \ \mathrm{t}(x) < T_-\} \end{cases} \tag{1}$$

has a unique solution depending continuously on h

Defining the scattering operator

We define the (geometric) scattering operator on $\widehat{N}$ by

$$\begin{aligned} &S : C_c^\infty(\mathcal{I}^-) \supset U \to C^\infty(\mathcal{I}^+), \\ &S(u\big|_{\mathcal{I}^-}) = u\big|_{\mathcal{I}^+}, \quad u = h \in U, \end{aligned}$$

for a neighbourhood U of the zero function in $C_c^\infty(\mathcal{I})$. Here u solves the (non-linear) scattering problem

$$\begin{cases} (\Box_g + B)u + Au^\kappa = 0, & \text{in } \{x \in \widehat{N} : \ \mathrm{t}(x) < T_+\}, \\ u\big|_{\mathcal{I}^-} = h, \\ u = 0 \text{ in } \{x \in \widehat{N} : \ \mathrm{t}(x) < T_-\} \end{cases} \tag{2}$$

Inverse scattering problem

Theorem (Alexakis, Isozaki, Lassas, T.)

If $\operatorname{supp}(a) = \mathbb{R}^{1+3}$*, the non-linear scattering operator* S*, defined in a neighborhood of the zero function in* $C_0^\infty(\mathcal{I}^-)$*, determines the conformal class of* g*.*
If $a \equiv 1$*, the conformal factor can also be determined.*

A non-physical extension

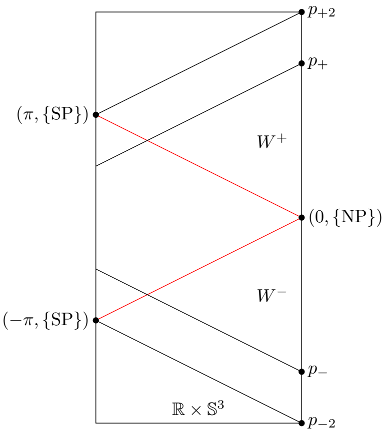
Assume that

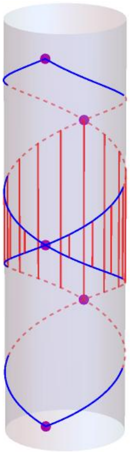
- the metric g is globally hyperbolic Lorentzian metric and $g_{ij} - \eta_{ij}$ belong to the Schwartz class
- $\widehat{g} := \Phi_* \widetilde{g}$ is the push-forward metric of $\widetilde{g} := \Omega^2 g$ on the Penrose diagram

Then $\widehat{g}$ can be smoothly extended to $\mathbb{R} \times \mathbb{S}^3$ by defining $g_e = \widehat{g}$ in $\widehat{N}$ and $g_e = \eta$, for $x \in \mathbb{R} \times \mathbb{S}^3 \setminus \widehat{N}$

Measurements beyond infinity

We can do an artificial extension of the Penrose diagram $\widehat{N}$ by gluing it into the cylinder $\mathbb{R} \times \mathbb{S}^3$:





Scattering operator determines a source-to-solution map

Given a source f supported in the non-physical past, we solve a linear wave equation

$$\begin{cases} (\Box_{g_e} + B)u = f, & \text{in } N_{\mathrm{ext}}, \\ \mathrm{supp}(u) \subset J^+(\mathrm{supp}(f)). \end{cases}$$

up to $\mathcal{I}^-$. Restricting u to $\mathcal{I}^-$, this is equivalent to

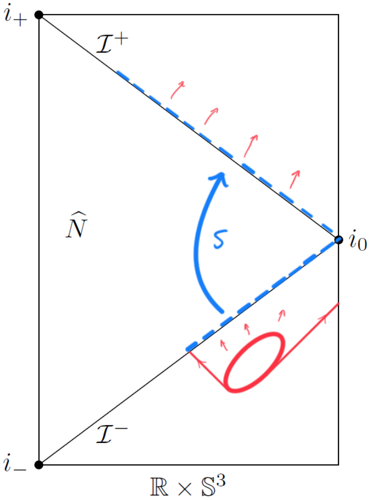
$$\begin{cases} (\Box_{g_e} + B)u = f, & \text{in } N_{\mathrm{ext}}, \\ u\big|_{\mathcal{I}^-} = h^-, \\ u(T_-, x) = \partial_t u(T_-, x) = 0, \end{cases}$$

which is a scattering problem in the Penrose diagram. (Here $T_- \leq \inf \mathrm{t}(\mathrm{supp}(\mathrm{f}))$).

Then the scattering operator determines $h^+ := u\big|_{\mathcal{I}^+} = S(u\big|_{\mathcal{I}^+})$. Finally, solving the linear Cauchy-Goursat problem

$$\begin{cases} (\Box_{g_e} + B)u = 0, & \text{in } N_{\text{ext}}, \\ u\big|_{\mathcal{I}^+} = h^+, \end{cases}$$

shows that we determine u in the non-physical future.



Rough sketch of the inverse scattering problem

- Scattering operator determines the source-to-solution map
- The source-to-solution map determines the scattering relation: **using the nonlinearity**
  - A. Feizmohammadi, M. Lassas, L. Oksanen: *Inverse problems for non-linear hyperbolic equations with disjoint sources and receivers.* Forum of Mathematics, Pi 9 (2021), Paper No. e10, 52

Higher order linearization

A k-fold linearization of the nonlinear equation

$$(\Box_g + B)u + Au^k = \sum_{j=1}^{k} \varepsilon_j f_j \tag{3}$$

with respect to ε_j yields

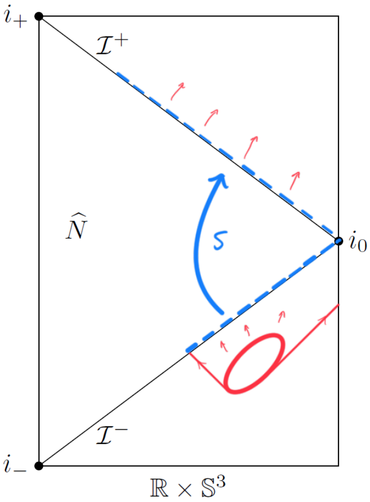
$$(\Box_g + B)w + Av_1 v_2 \cdots v_k = 0 \tag{4}$$

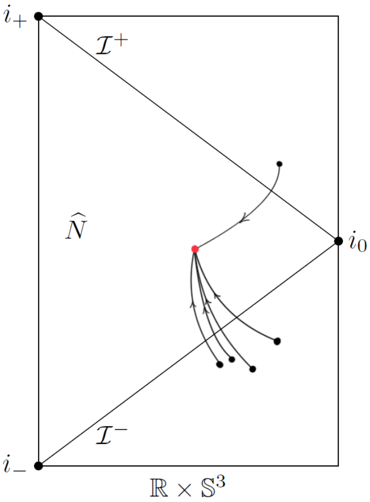
where

$$(\Box_g + B)v_j = f_j, \quad \text{in } \widehat{N}$$

The products $v_1 v_2 \cdots v_k$ can be used to produce point sources.

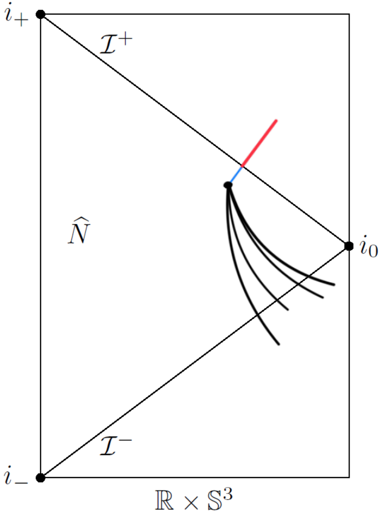
Scattering relation from the S-to-S map





Rough sketch of the inverse scattering problem

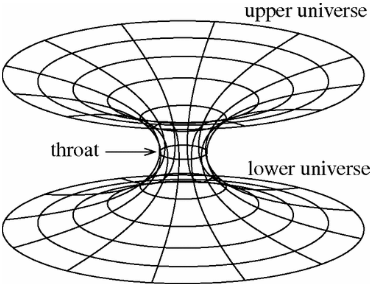
- The scattering relation determines the arrival time functions
- Arrival time functions determine light observation sets (and the differentiable structure of the manifold)

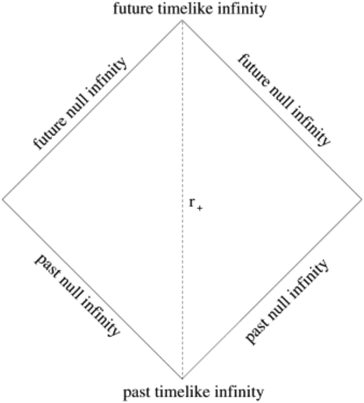


Rough sketch of the inverse scattering problem

- The light observation sets determine parts of lightcones, which themselves determine the full lightcones
- Knowledge of the lightcones determines the metric up to a conformal factor
- If the nonlinear term $A \equiv 1$, then one could also recover the conformal factor of the metric

More general manifolds can be allowed





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Thank you!